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Real space–time Green's functions applied to plate vibration induced by turbulent flow

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The paper describes calculations of the statistical properties of turbulence-induced plate vibration, the computational scheme making no reference to Fourier synthesis. The problem considered is that of a local turbulent field acting on an infinite thin elastic plate, and the statistics of the response field are described at large distances from the forcing region. Effects of mechanical dissipation in the plate are examined, along with a discussion of the relevance of the results to problems involving plates of finite extent.

1. Introduction

The equation describing the motion of a thin elastic plate under an externally applied pressure field $p(\mathbf{x},t)$ has the well-known form (e.g. Cremer & Heckl 1967),

$$\left(m\frac{\partial^2}{\partial t^2}+B\nabla^4\right)y(\mathbf{x},t)=p(\mathbf{x},t). \tag{1.1}$$

Here m is the mass of the plate per unit area, B the bending stiffness; y is the deflexion of the plate, and ∇^2 denotes the two-dimensional Laplacian. Mechanical dissipation in the plate is neglected for the moment. It is the object of this paper to present solutions of (1.1) giving the statistics of the plate response in terms of those of a random pressure field, and using entirely real space and time Green's functions. General results are obtained for the mean vibrational energy, the mean square bending moment and the mean energy flux generated at large distances from a finite region excited by a random, statistically steady pressure field. A knowledge of these three quantities is important, respectively, in problems of sound radiation from the plate, structural fatigue, and the motion of a finite plate excited by reverberant amplification of its normal modes. The particular case of plate excitation by a turbulent boundary layer is considered in detail. The dependence of the vibration parameters upon length and velocity scales characterizing the turbulent flow is obtained in the high and low speed limits, without any assumptions as to the detailed form of the pressure field statistics.

The analogous acoustic problem in three dimensions is governed by Lighthill's (1952) wave equation

$$\left(\frac{\partial^2}{\partial t^2}-a_0^2\nabla^2\right)\rho=\frac{\partial^2 T_{ij}}{\partial x_i\partial x_j}. \tag{1.2}$$

The turbulent excitation problem for this equation has been solved (Lighthill 1952, 1954; Ffowcs Williams 1963) without recourse to Fourier analysis. However, solution using only real space and time Green's functions is greatly facilitated by the fact that the Green's function for (1.2) contains a delta function

$$G(\mathbf{x},\mathbf{y},t,t^1) = \frac{\delta(t-t^1-|\mathbf{x}-\mathbf{y}|/a_0)}{4\pi a_0^2|\mathbf{x}-\mathbf{y}|}.$$

In two dimensions this is not so, but Ffowcs Williams & Hawkings (1968) have shown how the corresponding analysis can be carried through none the less. The physical problem used by these authors as a basis for a two-dimensional version of (1.2) is that of determining the wave amplitude in shallow water at large distances from a locally turbulent region. The problem to be considered here is more complicated, for waves in a plate form a dispersive system, while those on shallow water do not. A subsequent paper will use the methods given here to tackle the problem of dispersive water wave generation on water of great depth.

Now in previous discussions of (1.1) it has often been assumed that the random pressure field acts over the whole infinite surface of the plate, and that the pressure field is statistically homogeneous in space and stationary in time. Spectral analysis in space and time then allows the problem to be treated quite simply (e.g. Ffowcs Williams & Lyon 1963). However, the only mean value which is realistically given (in the sense that the results are relevant to any practical situation) by such a theory is the power input from the pressure field to the plate. Other mean values, such as the mean square velocity at any point, are infinite if mechanical dissipation is neglected (see Lighthill (1953) and Crighton (1969) for the analogous acoustic problem). Inclusion of finite dissipation leads to a finite intensity, but this does not make the results any more relevant. For first, the intensity is sensitive to the precise form of dissipation assumed, secondly the intensity is still so large that non-linear effects probably provide the proper control, rather than linear dissipative effects, and thirdly, we shall see that when only a small region of excitation is involved, the dissipative effects are in any case negligible compared with the radiation energy loss.

The non-singular vibration problem, involving only a finite excitation region, has been considered before, using Fourier time-analysis (Crighton 1968). The results obtained there will be recovered now, as a demonstration of the use of the space–time Green's functions in dispersive systems.

2. The Green's functions

The Green's function for (1.1) is essentially the solution of

$$\left(\frac{\partial^2}{\partial t^2}+\lambda^2\nabla^4\right)y = \frac{1}{m}\delta(\mathbf{x})\,\delta(t), \tag{2.1}$$

in which $\lambda = (B/m)^{\frac{1}{2}}$ has the dimensions of a kinematic diffusivity. The Green's function for the time-reduced (through the factor $\exp(i\omega t)$) plate equation is well known (Cremer & Heckl 1967, p. 258). It may be obtained from (2.1) by making a Fourier analysis into $(\mathbf{k}, \omega)$ space. The inverse integration over wave-number $\mathbf{k}$ is then performed, with due regard to a radiation condition as $|\mathbf{x}| \to \infty$. The complete Green's function is then found by an integration over frequency ω (Crighton 1968). One finds in this way the following solutions of (2.1):

$$v(\mathbf{x},t) = \frac{1}{4\pi m\lambda t}\sin\left(\frac{x^2}{4\lambda t}\right), \quad q(\mathbf{x},t) = \frac{-1}{4\pi Bt}\cos\left(\frac{x^2}{4\lambda t}\right). \tag{2.2}$$

Here $v = \partial y/\partial t$ is the velocity, while $q = \nabla^2 y$ is proportional to the bending moment. In terms of these quantities, the energy density is

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}Bq^2, \tag{2.3}$$

so that $\frac{1}{2}mv^2$ and $\frac{1}{2}Bq^2$ are the kinetic and elastic energies per unit area, respectively, and the energy flux vector is

$$\mathbf{F} = B(v\nabla q - q\nabla v). \tag{2.4}$$

The energy equation is readily derived from (1.1) in the form

$$\frac{\partial}{\partial t}\int_S E(\mathbf{x},t)\,d\mathbf{x} = \int_S p(\mathbf{x},t)\,v(\mathbf{x},t)\,d\mathbf{x} - \oint \mathbf{F}(\mathbf{x},t)\,.\,\boldsymbol{\nu}(\mathbf{x})\,dl(\mathbf{x}), \tag{2.5}$$

where the contour l, with unit outward normal $\boldsymbol{\nu}$, encloses the area S.

The Green's functions (2.2) have singularities which are associated with the fact that the functions exist only as generalized functions. Thus $v \not\to 0$ as $x = |\mathbf{x}| \to \infty$ for any $t > 0$, while $v \to \infty$ as $t \to 0+$ for any $x \neq 0$. The singular behaviour disappears when the excitation is distributed in space and time, and can also be made to disappear for the impulsive excitation if a suitable form for mechanical dissipation in the plate is postulated. To represent mechanical damping, one generally writes B in complex form, as $B(1+i\eta)$, where the constant η is known as the loss factor. Such a representation is obviously relevant only when a Fourier time analysis has been made. We suggest now the following modification of (1.1) to include dissipation:

$$\left(m\frac{\partial^2}{\partial t^2} - \lambda m\eta\nabla^2\frac{\partial}{\partial t} + B\nabla^4\right) y = p. \tag{2.6}$$

With this form of dissipation (the damping force being proportional to the time rate of change of the bending moment), a free flexural wave of frequency ω decays in amplitude like

$$\exp\left(-\frac{\omega\eta t}{2}\right).$$

This gives agreement with experiment if η is interpreted as the usual loss factor (for the experimental results on dissipation, see Cremer & Heckl 1967).

The Green's function for (2.6) can be shown to be

$$v(\mathbf{x},t) = \frac{1}{4\pi m\lambda t}\exp\left(-\frac{\eta}{2}\frac{x^2}{4\lambda t}\right)\sin\left(\frac{x^2}{4\lambda t} + \frac{\eta}{2}\right), \tag{2.7}$$

and, despite the impulsive excitation, this has none of the singular behaviour of (2.2). For the remainder of this paper we shall work with the solutions (2.2), though the rôle of mechanical damping will be discussed further in a subsequent section.

3. Plate response to a random pressure field

Suppose now that a random pressure field acts over a finite region of an infinite plate. The pressure field will be assumed stationary in time, and departures from spatial homogeneity over the finite region will be neglected. Then the pressure covariance

$$\langle p(\mathbf{y},t-\tau)p(\mathbf{y}+\mathbf{z},t-\tau^1)\rangle = P(\mathbf{z},\epsilon)$$

is an even function of the space separation $\mathbf{z}$ and of the time delay $\epsilon = \tau-\tau^1$, in the sense that $P(\mathbf{z},\epsilon) = P(-\mathbf{z},-\epsilon)$. $\langle\ldots\rangle$ here denotes an ensemble average, or equivalently, a time average.

The solution to (1.1) may be written, with the aid of (2.2), in the form

$$v(\mathbf{x},t) = \int_{\mathbf{y}}\int_{\tau=0}^{\infty} \frac{1}{4\pi m\lambda\tau}\sin\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4\lambda\tau}\right)p(\mathbf{y},t-\tau)\,d\mathbf{y}\,d\tau. \tag{3.1}$$

From this we form the mean value $\langle v^2(\mathbf{x})\rangle$, and consider an observation point $\mathbf{x}$ very distant from all points $\mathbf{y}$ at which the fluctuating pressure acts. As a first requirement, we take $|\mathbf{x}-\mathbf{y}|$ to be large compared with the correlation scale for the pressure, that being the distance $|\mathbf{z}|$ beyond which the spatial covariance $P(\mathbf{z},0)$ is effectively zero. We then find the following uniform (over the area of excitation) contribution to $\langle v^2(\mathbf{x})\rangle$ from unit area at $\mathbf{y}$;

$$\frac{\partial}{\partial\mathbf{y}}\langle v^2(\mathbf{x})\rangle = \iiint \frac{1}{(4\pi m\lambda)^2\tau\tau^1}\sin\left(\frac{x^2}{4\lambda\tau}\right)\sin\left(\frac{x^2-2\mathbf{x}.\mathbf{z}}{4\lambda\tau^1}\right)P(\mathbf{z},\tau-\tau^1)\,d\mathbf{z}\,d\tau\,d\tau^1. \tag{3.2}$$

Consider now the τ integral in (3.2). When x is large, contributions to the τ integral will be negligible because of cancellation, except for those large values of τ such that $\sin(x^2/4\lambda\tau)$ does *not* oscillate many times during the pressure correlation time scale (the value of ϵ beyond which the autocovariance $P(0,\epsilon)$ is negligibly small). Since we wish to apply this work to the boundary layer type of excitation, we ascribe a length scale L and a time scale L/U to the pressure covariance, L, U being typical length and velocity scales in the flow. Contributions to the τ integral then come from the τ for which

$$\frac{\partial}{\partial\tau}\left(\frac{x^2}{4\lambda\tau}\right) \lesssim \frac{U}{L},$$

i.e. for which

$$\tau \gtrsim (x/U)R^{\frac{1}{2}}.$$

Here $R = UL/\lambda$ is a kind of Reynolds number, based on the plate 'diffusivity' λ. The significance of R will be discussed in some detail later. Suppose now that we take $x \gg LR^{-\frac{1}{2}}$; we have already required that $x \gg L$, so that our condition on x is

$$x \gg \max(L, LR^{-\frac{1}{2}}). \tag{3.3}$$

Then all values of τ making a significant contribution to (3.2) are much larger than the time scale L/U of P. We may then write

$$\tau - \tau^1 = \epsilon, \quad \frac{1}{\tau^1} = \frac{1}{\tau} + \frac{\epsilon}{\tau^2},$$

the remaining terms being negligible. Further, the limits of the ϵ integration can be taken as $\pm\infty$, and (3.2) reduces to

$$\frac{\partial}{\partial \mathbf{y}}\langle v^2(\mathbf{x})\rangle = \int_{\mathbf{z}}\int_{\tau=0}^{\infty}\int_{\epsilon=-\infty}^{+\infty} \frac{1}{(4\pi m\lambda)^2}\frac{1}{\tau^2}\sin\left(\frac{x^2}{4\lambda\tau}\right) \times \sin\left\{\frac{x^2 - 2\mathbf{x}.\mathbf{z} + \epsilon x^2/\tau}{4\lambda\tau}\right\} P(\mathbf{z},\epsilon)\,d\mathbf{z}\,d\tau\,d\epsilon. \tag{3.4}$$

The τ integral can now be performed exactly, and the result contains three kinds of term. The first terms, quoted below, are even functions of $(\mathbf{z},\epsilon)$, of order x^{-1} as $x \to \infty$. Secondly, one has odd functions of ϵ (Fresnel integrals) which do not contribute to the $(\mathbf{z},\epsilon)$ integral. Finally, there are Fresnel integrals whose asymptotic forms, for fixed ϵ and $x \to \infty$, are of order $x^{-\frac{3}{2}}$, and these may be discarded. Thus we find

$$\frac{\partial}{\partial \mathbf{y}}\langle v^2(\mathbf{x})\rangle = \int_{\mathbf{z}}\int_{\epsilon=0}^{\infty} \sqrt{\left(\frac{\pi\lambda}{\epsilon}\right)} \frac{1}{(4\pi m\lambda)^2 x}\cos\left(\frac{\pi}{4} - \frac{(\mathbf{x}.\mathbf{z})^2}{4\lambda\epsilon x^2}\right) P(\mathbf{z},\epsilon)\,d\mathbf{z}\,d\epsilon. \tag{3.5}$$

A similar calculation for the mean square of q gives simply

$$B\langle q^2(\mathbf{x})\rangle = m\langle v^2(\mathbf{x})\rangle, \tag{3.6}$$

so that the time averaged kinetic and elastic energies at any point are equal, as one would expect.

The calculation of the energy flux vector from the definition (2.4) is tedious. It is sufficient, as $x \to \infty$, to calculate the component F_x of $\mathbf{F}$ in the radial direction from any convenient origin in the excitation region. The integrated energy flux around a distant contour l then gives the rate of working of the pressure over the excitation area, for the averaged form of (2.5) may be written as

$$\frac{\partial}{\partial \mathbf{y}}\oint \langle F_x\rangle x\,d\theta = \langle p(\mathbf{y},t)\,v(\mathbf{y},t)\rangle. \tag{3.7}$$

We find that

$$\frac{\partial}{\partial \mathbf{y}}\langle F_x(\mathbf{x})\rangle = \int_{\mathbf{z}}\int_{\epsilon=0}^{\infty} \frac{1}{2^{\frac{5}{2}} m(\pi\lambda)^{\frac{3}{2}}}\frac{|\mathbf{x}.\mathbf{z}|}{\epsilon^{\frac{3}{2}} x^2}\, G\left(\frac{(\mathbf{x}.\mathbf{z})^2}{4\lambda\epsilon x^2}\right) P(\mathbf{z},\epsilon)\,d\mathbf{z}\,d\epsilon,$$

where $G(\mu) = \sin\mu\, S_2(\mu) + \cos\mu\, C_2(\mu)$, and C_2, S_2 denote the Fresnel integrals

$$C_2(\mu) = \int_0^{\mu} \frac{\cos t}{(2\pi t)^{\frac{1}{2}}}\,dt, \quad S_2(\mu) = \int_0^{\mu} \frac{\sin t}{(2\pi t)^{\frac{1}{2}}}\,dt.$$

Writing $\mathbf{x}.\mathbf{z} = xz\cos\theta$, the integrated energy flux is found as an integral over θ, so that from (3.7),

$$\langle p(\mathbf{y})\,v(\mathbf{y})\rangle = \int_{z=0}^{\infty}\int_{\theta=0}^{2\pi}\int_{\epsilon=0}^{\infty} \frac{1}{2^{\frac{3}{2}}(\pi\lambda)^{\frac{3}{2}} m}\left(\frac{z\cos\theta}{\epsilon^{\frac{3}{2}}}\right) P(z,\theta,\epsilon)\, G\left(\frac{z^2\cos^2\theta}{4\lambda\epsilon}\right) z\,dz\,d\theta\,d\epsilon. \tag{3.8}$$

For future use, note that when μ is small,

$$S_2(\mu) = \frac{1}{(2\pi)^{\frac{1}{2}}}\tfrac{2}{3}\mu^{\frac{3}{2}} + O(\mu^{\frac{5}{2}}), \quad C_2(\mu) = \frac{1}{(2\pi)^{\frac{1}{2}}}2\mu^{\frac{1}{2}} + O(\mu^{\frac{3}{2}}).$$

The formulae (3.5), (3.6) and (3.8) give expressions for the far-field energies and the power input of the pressure field in terms of weighted integrals of the pressure covariance. In the next section, these formulae will be expressed in the non-dimensional form appropriate to boundary layer excitation, and the dependence of the vibration intensities upon the plate and turbulence parameters will be obtained.

4. Excitation by boundary layer turbulence

We suppose now that fluid flows over the plate in turbulent motion. When the flow Reynolds number is high and the Mach number low, the flow may be characterized in the usual way by length and velocity scales L, U, for example, the boundary layer thickness and the free stream velocity. If ρ is the (constant) fluid density, we define dimensionless variables according to

$$\mathbf{z} = \hat{\mathbf{z}}L, \quad \epsilon = \hat{\epsilon}L/U, \quad P(\mathbf{z},\epsilon) = \rho^2 U^4 \hat{P}(\hat{\mathbf{z}},\hat{\epsilon}). \tag{4.1}$$

The parameter $R = UL/\lambda$ is defined as before. Further, we take a quasi-Cartesian set of axes for the far-field, z_x representing the co-ordinate in the direction $\mathbf{x}$, z_y the perpendicular co-ordinate. It is convenient also to take $u = \hat{z}_x/\hat{\epsilon}^{\frac{1}{2}}$ as a new variable in place of $\hat{z}_x$. Then, dropping the caret signs, and discarding numerical constants which may very easily be recovered, we find from (3.5),

$$\frac{\partial}{\partial \mathbf{y}}\langle v^2(\mathbf{x})\rangle \sim R^{\frac{3}{2}}\left(\frac{U}{L}\right)^2\left(\frac{\rho L}{m}\right)^2\left(\frac{L}{x}\right)\int \cos\left(\frac{\pi}{4}-\frac{u^2R}{4}\right)P(u\epsilon^{\frac{1}{2}},z_y,\epsilon)\,du\,dz_y\,d\epsilon. \tag{4.2}$$

If we now let $R \to 0$ in the integral, it becomes

$$\int P(u\epsilon^{\frac{1}{2}},z_y,\epsilon)\,du\,dz_y\,d\epsilon.$$

However, this results in a poor estimate for $R \ll 1$, for in strictly incompressible flow, the integrated covariance

$$\int P(\mathbf{z},\epsilon)\,d\mathbf{z}$$

is zero, while in slightly compressible flow it is of order (Mach number)^2 (Ffowcs Williams 1965). It is therefore necessary to take the next term of the expansion of the cosine in R, and then the integral becomes

$$R\int u^2 P(u\epsilon^{\frac{1}{2}},z_y,\epsilon)\,du\,dz_y\,d\epsilon.$$

Apart from the factor R, this expression is a function entirely of the direction (θ) of the vector $\mathbf{x}-\mathbf{y}$, so that

$$\frac{\partial}{\partial \mathbf{y}}\langle v^2(\mathbf{x})\rangle \sim R^{\frac{5}{2}}\left(\frac{U}{L}\right)^2\left(\frac{\rho L}{m}\right)^2\left(\frac{L}{x}\right)f(\theta) \quad (R \ll 1). \tag{4.3}$$

In the case $R \gg 1$, we rescale u according to $u = \alpha R^{-\frac{1}{2}}$, and then the integral in (4.2) becomes

$$\int \left(\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right) P\left(\frac{\alpha\epsilon^{\frac{1}{2}}}{R^{\frac{1}{2}}}, z_y, \epsilon\right) \frac{d\alpha}{R^{\frac{1}{2}}} dz_y\, d\epsilon.$$

Letting $R \to \infty$, the asymptotic value of this integral is

$$R^{-\frac{1}{2}} \int \left(\cos\frac{\alpha^2}{4} + \sin\frac{\alpha^2}{4}\right) P(0, z_y, \epsilon)\, d\alpha\, dz_y\, d\epsilon,$$

and again, apart from the factor $R^{-\frac{1}{2}}$, this is a function of direction only. Hence

$$\frac{\partial}{\partial \mathbf{y}} \langle v^2(\mathbf{x})\rangle \sim R\left(\frac{U}{L}\right)^2 \left(\frac{\rho L}{m}\right)^2 \left(\frac{L}{x}\right) g(\theta) \quad (R \gg 1). \tag{4.4}$$

In this high-speed limit, the dimensionless energy varies directly as R.

The results for the power input are obtained in a similar fashion. For $R \ll 1$ we have to use the expansions of the Fresnel functions quoted earlier, and thus obtain

$$\langle pv\rangle \sim R^2(\rho U^3)\left(\frac{\rho L}{m}\right) \quad (R \ll 1). \tag{4.5}$$

When $R \gg 1$ we need a slightly different rescaling from that used above. The angular integration of the pressure covariance has already been performed as a consequence of the integration of the energy flux round a distant contour, so that the appropriate rescaling is now $z = \alpha R^{-\frac{1}{2}}$, $z\,dz = \alpha\, d\alpha\, R^{-1}$. This gives us now

$$\langle pv\rangle \sim (\rho U^3)\left(\frac{\rho L}{m}\right) \quad (R \gg 1). \tag{4.6}$$

Defining an efficiency as $\gamma = \langle pv\rangle/\rho U^3$, a measure of the rate of energy loss into the plate compared with the rate of advection of energy in the flow, we see that $\gamma \sim R^2$ when R is small, and that γ asymptotes to a constant value as $R \to \infty$. These results, and also those for the energy, are exactly those found by Fourier time analysis (Crighton 1968). Further details of the plate vibration, for example the directivity pattern, can be worked out if measured forms of the space–time pressure covariance are available. Effects of convection of the pressure field could also be examined, though we have not been able to obtain any useful results for those effects. It is probably better to examine properties arising from source convection in terms of Fourier frequency components, where one has a clear distinction between convection velocities which are 'subsonic' and 'supersonic' relative to the free wave speed at any given frequency. This distinction disappears in real time from a strongly dispersive system.

5. Discussion

It is interesting to interpret the parameter $R^{\frac{1}{2}}$ as a Mach number, in analogy with the aerodynamic noise problem. The results obtained above can be interpreted in terms of certain integrals of the pressure wave-number frequency spectrum in the Fourier treatment of the problem. The spectrum is evaluated at the wave-number $k = (\omega/\lambda)^{\frac{1}{2}}$ corresponding to a free flexural wave of frequency ω, and is then integrated over frequency. When R is small, i.e. λ is large, the integration path $\omega = \lambda k^2$ lies close to the ω axis of the (k, ω) plane, and therefore excitation and response have the *same typical frequency* U/L. The response wavelength is then $\Lambda = LR^{-\frac{1}{2}}$, so that when $R^{\frac{1}{2}} \ll 1$, turbulence scales are small compared with the radiated wavelength. Note that condition (3.3) is then $x \gg \Lambda$, which is the far-field condition with respect to the typical wavelength Λ.

Suppose now that $R \gg 1$, i.e. λ is small. Then the curve $\omega = \lambda k^2$ lies very close to the k axis, and hence *length scales* in both excitation and response are equal. On the other hand, the time scale of the response is now RL/U, and this is large compared with the excitation time scale L/U. Since Λ and L are now equal, condition (3.3) is still the far-field condition for wavelength Λ. Similar considerations apply in the acoustic problem, where one has $\omega = a_0 k$ in place of $\omega = \lambda k^2$, and the Mach number $M = U/a_0$ in place of $R^{\frac{1}{2}}$ (or in place of R, for the quadratic dispersion relation $\omega = \lambda k^2$ allows both as a suitable kind of Mach number). There the cases $M \ll 1$, $M \gg 1$ are referred to as the 'acoustically compact' and 'non-compact' limits.

These ideas emerge also from the space–time analysis. We have seen already that contributions to the intensity come only from the x, τ satisfying

$$\frac{\partial}{\partial \tau}\left(\frac{x^2}{\lambda \tau}\right) \lesssim \frac{U}{L}, \quad \text{or} \quad \frac{x}{\tau} \lesssim UR^{-\frac{1}{2}}.$$

Similar consideration of (3.5) and (3.8) shows that this requirement must also be satisfied when x, τ are replaced by the *separation* variables z, ϵ:

$$\frac{z}{\epsilon} \lesssim UR^{-\frac{1}{2}}. \tag{5.1}$$

This condition may be referred to as a time-matching; the trigonometric terms in (3.5) or (3.8) must *not* oscillate rapidly with ϵ on the time scale L/U of the pressure fluctuations. There is another condition which must be satisfied if we are to obtain an appreciable contribution to the intensity. This is the space-matching

$$\frac{\partial}{\partial z}\left(\frac{z^2}{\lambda \epsilon}\right) \lesssim \frac{1}{L},$$

stating that trigonometric terms must not oscillate rapidly with z on the length scale L of the pressure fluctuations. Thus

$$\frac{z}{\epsilon} \lesssim UR^{-1} \text{ for the space-matching,} \tag{5.2}$$

and (5.1) and (5.2) must be satisfied simultaneously. Clearly, when $R^{\frac{1}{2}} \ll 1$ it is sufficient to match the time scales, while when $R^{\frac{1}{2}} \gg 1$ the length scales must be matched, which is in accordance with the ideas given earlier.

We can now give a criterion for a finite plate, of dimension D, to be effectively infinite. When $R^{\frac{1}{2}} \ll 1$, the z, ϵ making appreciable contributions to the intensity satisfy $z/\epsilon \lesssim UR^{-\frac{1}{2}}$, so that $c = UR^{-\frac{1}{2}}$ is an upper limit to the effective speed with which waves in the plate travel. (Alternatively, c is given by $c = \lambda k$ with the typical response wave-number $k = L^{-1}R^{\frac{1}{2}}$.) We can regard the plate as infinite if the time taken by a wave travelling at speed c to cross the distance D is large compared with the time scale L/U of the source. Thus we require

$$\frac{D}{c} \gg \frac{L}{U} \quad \text{or} \quad D \gg \Lambda.$$

When $R^{\frac{1}{2}} \gg 1$ we require similarly $D \gg LR^{-1}$, and since the spatial condition $D \gg L$ is obviously also necessary, we find again that

$$D \gg \Lambda = LR^{-\frac{1}{2}} \quad \text{or} \quad L \tag{5.3}$$

for the plate to be effectively infinite.

In underwater applications it appears that R is generally small ($R \sim 10^{-2}$ is perhaps typical), so that plates of typical dimension $10L$ may be regarded as infinite. In aeronautical contexts, R is usually much larger, and the condition (5.3) is more easily attained.

We return now to the question of mechanical dissipation. With the dissipation represented by (2.7) it is easily seen that the attenuation length l is given by

$$l = \eta^{-1}\Lambda, \tag{5.4}$$

where the wavelength $\Lambda = LR^{-\frac{1}{2}}$ for $R^{\frac{1}{2}} \ll 1$, and $\Lambda = L$ for $R^{\frac{1}{2}} \gg 1$. We can therefore satisfy the far-field condition (3.3), with neglect of mechanical dissipation, for values of x satisfying

$$\Lambda \ll x \ll \eta^{-1}\Lambda. \tag{5.5}$$

Since the loss factor is generally of order 10^{-2}, condition (5.5) is not very stringent, and dissipation is negligible except at extreme distances.

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